

# A SPECIAL CASE OF THE TWO-DIMENSIONAL JACOBIAN CONJECTURE

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**ABSTRACT.** Let  $f : \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y]$  be a  $\mathbb{C}$ -algebra endomorphism having an invertible Jacobian.

We show that for such  $f$ , if, in addition, the group of invertible elements of  $\mathbb{C}[f(x), f(y), x][1/v] \subset \mathbb{C}(x, y)$  is contained in  $\mathbb{C}(f(x), f(y)) - 0$ , then  $f$  is an automorphism. Here  $v \in \mathbb{C}[f(x), f(y)] - 0$  is such that  $y = u/v$ , with  $u \in \mathbb{C}[f(x), f(y), x] - 0$ .

Keller's theorem (in dimension two) follows immediately, since Keller's condition  $\mathbb{C}(f(x), f(y)) = \mathbb{C}(x, y)$  implies that the group of invertible elements of  $\mathbb{C}[f(x), f(y), x][1/v]$  is contained in  $\mathbb{C}(x, y) - 0 = \mathbb{C}(f(x), f(y)) - 0$ .

## 1 Introduction

Throughout this note,  $f : \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y]$  is a  $\mathbb{C}$ -algebra endomorphism that satisfies  $\text{Jac}(p, q) \in \mathbb{C}^*$ , where  $p := f(x)$  and  $q := f(y)$ .

Formanek's field of fractions theorem [6, Theorem 2] in dimension two says that  $\mathbb{C}(p, q, x) = \mathbb{C}(x, y)$ . From this it is not difficult to obtain that  $y = u/v$ , for some  $u \in \mathbb{C}[p, q, x] - 0$  and  $v \in \mathbb{C}[p, q] - 0$ .

We show in Theorem 3.1 that for such  $f$ , if, in addition, the group of invertible elements of  $\mathbb{C}[p, q, x][1/v]$  is contained in  $\mathbb{C}(p, q) - 0$ , then  $f$  is an automorphism.

Our proof of Theorem 3.1 is almost identical to the proof of Formanek's automorphism theorem [5, Theorem 1]; we did only some slight changes in his proof, and also used Formanek's field of fractions theorem and Wang's intersection theorem [15, Theorem 41 (i)].

Keller's theorem in dimension two follows immediately from our theorem: Assume that  $\mathbb{C}(p, q) = \mathbb{C}(x, y)$ . Then our condition of Theorem 3.1 is satisfied, because the group of invertible elements of  $\mathbb{C}[p, q, x][1/v] \subset \mathbb{C}(x, y)$  is contained in  $\mathbb{C}(x, y) - 0 = \mathbb{C}(p, q) - 0$ .

## 2 Preliminaries

Our Theorem 3.1 deals with the two-dimensional case only. However, the results we rely on are valid in any dimension  $n$ , so we add the following notation:  $F : \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{C}[x_1, \dots, x_n]$  is a  $\mathbb{C}$ -algebra endomorphism that satisfies  $\text{Jac}(F_1, \dots, F_n) \in \mathbb{C}^*$ , where  $F_1 := F(x_1), \dots, F_n := F(x_n)$ . When  $n = 2$  we will keep the above notation, namely,  $x_1 = x, x_2 = y, F_1 = p, F_2 = q$ .

**Theorem 2.1** (Formanek's automorphism theorem). *Suppose that there is a polynomial  $W$  in  $\mathbb{C}[x_1, \dots, x_n]$  such that  $\mathbb{C}[F_1, \dots, F_n, W] = \mathbb{C}[x_1, \dots, x_n]$ . Then  $\mathbb{C}[F_1, \dots, F_n] = \mathbb{C}[x_1, \dots, x_n]$ , namely,  $F$  is an automorphism.*

*Proof.* See [5, Theorem 1] and [4, page 13, Exercise 9]. □

- If there exists  $w \in \mathbb{C}[x, y]$  such that  $\mathbb{C}[p, q, w] = \mathbb{C}[x, y]$ , then  $\mathbb{C}[p, q] = \mathbb{C}[x, y]$ , namely,  $f$  is an automorphism.

**Theorem 2.2** (Formanek's field of fractions theorem).

$$\mathbb{C}(F_1, \dots, F_n, x_1, \dots, x_{n-1}) = \mathbb{C}(x_1, \dots, x_n).$$

*Proof.* See [6, Theorem 2]. □

- $\mathbb{C}(p, q, x) = \mathbb{C}(x, y)$  and  $\mathbb{C}(p, q, y) = \mathbb{C}(x, y)$ .

Formanek remarks that when  $n = 2$ ,  $\mathbb{C}(p, q, w) = \mathbb{C}(x, y)$ , where  $w$  is the image of  $x$  under any automorphism of  $\mathbb{C}[x, y]$ ; see [6, page 370, just before Theorem 6].

The two-dimensional case was already proved by Moh [12, page 151] and by Hamann [7, Lemma 2.1, Proposition 2.1(2)]. Moh and Hamann assumed that  $p$  is monic in  $y$ , but this is really not a restriction.

It is easy to see that:

**Corollary 2.3.** *There exist  $u \in \mathbb{C}[p, q, x] - 0$  and  $v \in \mathbb{C}[p, q] - 0$  such that  $y = u/v$ .*

*Proof.*  $y \in \mathbb{C}(x, y) = \mathbb{C}(p, q, x) = \mathbb{C}(p, q)(x)$ . Since  $x$  is algebraic over  $\mathbb{C}(p, q)$ , we have  $\mathbb{C}(p, q)(x) = \mathbb{C}(p, q)[x]$  (see [14, Remark 4.7]). Hence,  $y \in \mathbb{C}(p, q)[x]$ . Therefore, there exist  $a_i, b_i \in \mathbb{C}[p, q]$  ( $b_i \neq 0$ ) such that  $y = \sum (a_i/b_i)x^i$ . Then if we denote  $B = \prod b_i$  and  $B_i$  the product of the  $b_j$ 's except  $b_i$ , we get  $y = (1/B) \sum B_i a_i x^i$ . Just take  $v := B$  and  $u := \sum B_i a_i x^i$ . □

**Theorem 2.4** (Wang's intersection theorem).  $\mathbb{C}(F_1, \dots, F_n) \cap \mathbb{C}[x_1, \dots, x_n] = \mathbb{C}[F_1, \dots, F_n]$ .

*Proof.* See [15, Theorem 41 (i)] and [4, Corollary 1.1.34 (ii)]. □

Wang's intersection theorem has a more general version due to Bass [2, Remark after Corollary 1.3, page 74], [4, Proposition D.1.7]; we will not need the more general version here.

- $\mathbb{C}(p, q) \cap \mathbb{C}[x, y] = \mathbb{C}[p, q]$ .

The following is immediate:

**Corollary 2.5.**  $\mathbb{C}(p, q) \cap R = \mathbb{C}[p, q]$ , for any  $\mathbb{C}[p, q] \subseteq R \subseteq \mathbb{C}[x, y]$ . In particular,  $\mathbb{C}(p, q) \cap \mathbb{C}[p, q, x] = \mathbb{C}[p, q]$ .

*Proof.*  $\mathbb{C}(p, q) \cap R \subseteq \mathbb{C}(p, q) \cap \mathbb{C}[x, y] = \mathbb{C}[p, q]$ . The other inclusion,  $\mathbb{C}(p, q) \cap R \supseteq \mathbb{C}[p, q]$ , is trivial. □

**Theorem 2.6** (Keller's theorem). *If  $\mathbb{C}(F_1, \dots, F_n) = \mathbb{C}(x_1, \dots, x_n)$ , then  $F$  is an automorphism.*

$F$  as in Keller's theorem is called birational ( $F$  has an inverse formed of rational functions).

*Proof.* See [9], [4, Corollary 1.1.35] and [1, Theorem 2.1]. □

- If  $\mathbb{C}(p, q) = \mathbb{C}(x, y)$ , then  $f$  is an automorphism.

*Remark 2.7.* Notice that the above results are dealing with  $k[x_1, \dots, x_n]$ , where  $k$  is:

- $\mathbb{C}$ : Formanek's field of fractions theorem.
- a field of characteristic zero: Formanek's automorphism theorem.
- any field: Keller's theorem.
- a UFD: Wang's intersection theorem.

We have not checked if Formanek's field of fractions theorem is valid over a more general field than  $\mathbb{C}$ ; if, for example, it is valid over any algebraic closed field of characteristic zero, then our Theorem 3.1 is valid over any algebraic closed field of characteristic zero, not just over  $\mathbb{C}$ .

Anyway, working over  $\mathbb{C}$  is good enough in view of [4, Lemma 1.1.14].

### 3 A new proof of Keller's theorem in dimension two

Our proof of Theorem 3.1 relies heavily on the proof of Formanek's automorphism theorem; we did only some slight changes in his proof, changes that seem quite natural in view of Corollary 2.3:

Although we do not know if  $\mathbb{C}[p, q, x] = \mathbb{C}[x, y]$  (if so, then  $f$  is an automorphism by Formanek's automorphism theorem), we do know that  $\mathbb{C}(p, q, x) = \mathbb{C}(x, y)$  (by Formanek's field of fractions theorem), so by Corollary 2.3,  $y = u/v$  for some  $u \in \mathbb{C}[p, q, x] - 0$  and  $v \in \mathbb{C}[p, q] - 0$ . Therefore, it seems natural to consider  $\beta : \mathbb{C}[U_1, U_2, U_3][1/V] \rightarrow \mathbb{C}[p, q, x][1/v]$ , where  $V = v(U_1, U_2)$ .

This  $\beta$  has  $x$  and  $y$  in its image, so most of Formanek's proof can be adjusted here, except that the group of invertible elements of  $\mathbb{C}[p, q, x][1/v]$  is not as easily described as the group of invertible elements of  $\mathbb{C}[x, y]$ , which is obviously  $\mathbb{C}^*$ .

Only after adding a condition on the group of invertible elements of  $\mathbb{C}[p, q, x][1/v]$ , we are able to show that  $f$  is an automorphism.

Now we are ready to bring our theorem; we recommend the reader to first read the proof of Formanek's automorphism theorem, and then read our proof, with  $p, q, x$  in our proof instead of  $F_1, F_2, F_3$  in his proof.

**Theorem 3.1** (Main Theorem). *If the group of invertible elements of  $\mathbb{C}[p, q, x][1/v]$  is contained in  $\mathbb{C}(p, q) - 0$ , then  $f$  is an automorphism.*

*Proof.* By Corollary 2.3, there exist  $u \in \mathbb{C}[p, q, x] - 0$  and  $v \in \mathbb{C}[p, q] - 0$  such that  $y = u/v$ .

Let  $U_1, U_2, U_3$  be independent variables over  $\mathbb{C}$ . Define  $\alpha : \mathbb{C}[U_1, U_2, U_3] \rightarrow \mathbb{C}[p, q, x]$  by  $\alpha(U_1) := p$ ,  $\alpha(U_2) := q$ ,  $\alpha(U_3) := x$ . Clearly,  $\alpha$  is surjective.

Claim: The kernel of  $\alpha$  is a principal prime ideal of  $\mathbb{C}[U_1, U_2, U_3]$ .

Proof of claim:  $\mathbb{C}(U_1, U_2, U_3)$  has transcendence degree 3 over  $\mathbb{C}$ , and  $\mathbb{C}(p, q, x) = \mathbb{C}(x, y)$  has transcendence degree 2 over  $\mathbb{C}$ . From [11, Theorem 5.6],  $\mathbb{C}[U_1, U_2, U_3]$  is of Krull dimension 3 and  $\mathbb{C}[p, q, x]$  is of Krull dimension 2. Hence, the kernel of  $\alpha$  is of height 1, and in a Noetherian UFD a height one prime ideal is principal, see [3, Theorem 15.9].

Denote by  $H$  a generator of the kernel of  $\alpha$ :  $H = H_r U_3^r + \dots + H_1 U_3 + H_0$ , where  $H_j \in \mathbb{C}[U_1, U_2]$  and  $r \geq 1$ .  $H$  is a product of the minimal polynomial for  $x$  over  $\mathbb{C}(p, q)$  by some element  $H_r$  of  $\mathbb{C}[U_1, U_2]$  which clears the denominators of the minimal polynomial for  $x$  over  $\mathbb{C}(p, q)$ . Notice that  $r = 0$  is impossible, since then  $H = H_0(U_1, U_2)$ :

- If  $H_0(U_1, U_2) \equiv 0$ , then  $H(U_1, U_2, U_3) \equiv 0$ , so the kernel of  $\alpha$  is zero, but then we have  $\mathbb{C}[U_1, U_2, U_3] \cong \mathbb{C}[p, q, x]$ , which is impossible from considerations of Krull dimensions.
- If  $H_0(U_1, U_2) \neq 0$ , then  $0 = \alpha(H) = \alpha(H_0(U_1, U_2)) = H_0(p, q)$  is a non-trivial algebraic dependence of  $p$  and  $q$  over  $\mathbb{C}$ . But  $p$  and  $q$  are algebraically independent over  $\mathbb{C}$ , because  $\text{Jac}(p, q) \neq 0$ ; see [10, pages 19-20] or [14, Proposition 6A.4].

Since we do not know if  $y$  is in the image of  $\alpha$ , we define the following (surjective)  $\beta : \mathbb{C}[U_1, U_2, U_3][1/V] \rightarrow \mathbb{C}[p, q, x][1/v]$  by  $\beta(U_1) := p$ ,  $\beta(U_2) := q$ ,  $\beta(U_3) := x$ ,

$\beta(1/V) := 1/(\beta(V))$ , where  $V := v(U_1, U_2)$ , namely, in  $v \in \mathbb{C}[p, q] - 0$  replace  $p$  by  $U_1$  and  $q$  by  $U_2$  and get  $V$ . It is clear that  $\beta(V) = v$ , so  $\beta(1/V) = 1/v$ .

Notice that  $V \in \mathbb{C}[U_1, U_2]$ ; the fact that the  $U_3$ -degree of  $V$  is zero will be crucial in what follows.

Now,  $y$  is in the image of  $\beta$ ; indeed, let  $U := u(U_1, U_2, U_3)$ , namely, in  $u \in \mathbb{C}[p, q, x] - 0$  replace  $p$  by  $U_1$ ,  $q$  by  $U_2$  and  $x$  by  $U_3$ , and get  $U$ . Then clearly  $\beta(U/V) = u/v = y$ .

Take:  $T_1 := U_3$  and  $T_2 := U/V$ . Then,  $\beta(T_1) = \beta(U_3) = x$ , and  $\beta(T_2) = \beta(U/V) = u/v = y$ .

Each of the following three elements lie in the kernel of  $\beta$ :  $U_1 - p(T_1, T_2)$ ,  $U_2 - q(T_1, T_2)$  and  $U_3 - x(T_1, T_2) = U_3 - T_1 = 0$ . Indeed,  $\beta(U_1 - p(T_1, T_2)) = \beta(U_1) - \beta(p(T_1, T_2)) = p - p = 0$  and  $\beta(U_2 - q(T_1, T_2)) = \beta(U_2) - \beta(q(T_1, T_2)) = q - q = 0$ .

Claim: The kernel of  $\beta$  is a principal prime ideal of  $\mathbb{C}[U_1, U_2, U_3][1/V]$ , generated by exactly the same  $H \in \mathbb{C}[U_1, U_2, U_3]$  that generates the kernel of  $\alpha$ .

Proof of claim: Assume that  $R/V^j$  is in the kernel of  $\beta$ , where  $R \in \mathbb{C}[U_1, U_2, U_3]$ . We have  $0 = \beta(R/V^j) = \beta(R)/\beta(V)^j = \beta(R)/v^j$ , hence  $0 = \beta(R)$ . Since  $\beta$  restricted to  $\mathbb{C}[U_1, U_2, U_3]$  is  $\alpha$ , we get that  $R$  belongs to the kernel of  $\alpha$ , hence  $R = \tilde{R}H$ , for some  $\tilde{R} \in \mathbb{C}[U_1, U_2, U_3]$ . So,  $R/V^j = \tilde{R}H/V^j = (\tilde{R}/V^j)H$ , as claimed.

Therefore, there exist  $R_1, R_2 \in \mathbb{C}[U_1, U_2, U_3]$  ( $R_3 = 0$ ) and  $n, m \geq 0$  such that  $U_1 - p(T_1, T_2) = (R_1/V^n)H$  and  $U_2 - q(T_1, T_2) = (R_2/V^m)H$ . So,  $U_1 = p(T_1, T_2) + (R_1/V^n)H$  and  $U_2 = q(T_1, T_2) + (R_2/V^m)H$  (and  $U_3 = T_1$ ).

Differentiating these three equations with respect to  $U_1, U_2, U_3$  and using the Chain Rule, we get similar matrices to those in Formanek's proof; the difference is that instead of  $R_1, R_2, R_3$  of Formanek's proof, we have here  $R_1/V^n, R_2/V^m, 0$ .

Applying  $\beta$  gives a matrix equation over  $\mathbb{C}[p, q, x][1/v]$ , similar to the matrix equation (2) of Formanek's proof.

Cramer's Rule shows that  $\beta(\partial H / \partial U_3) = \lambda/d$ , where  $\lambda = \text{Jac}(p, q) \in \mathbb{C}^*$  and  $d \in \mathbb{C}[p, q, x][1/v] - 0$  is the determinant of the matrix on the left.

$d$  belongs to the group of invertible elements of  $\mathbb{C}[p, q, x][1/v]$ , hence, by our assumption,  $d$  belongs to  $\mathbb{C}(p, q) - 0$ .

On the one hand,  $d \in \mathbb{C}[p, q, x][1/v] - 0$ , hence  $d = \tilde{d}/v^l$  for some  $\tilde{d} \in \mathbb{C}[p, q, x] - 0$  and  $l \geq 0$ . On the other hand,  $d \in \mathbb{C}(p, q) - 0$ , hence  $d = a/b$  for some  $a, b \in \mathbb{C}[p, q] - 0$ . Combining the two we get,  $\tilde{d}/v^l = a/b$ , so  $\mathbb{C}[p, q, x] - 0 \ni \tilde{d} = v^l(a/b) \in \mathbb{C}(p, q) - 0$ . From Corollary 2.5 we get that  $\tilde{d} \in \mathbb{C}[p, q] - 0$ .

(Remark: Actually, one can use Wang's intersection theorem directly, without Corollary 2.5, and still get  $\tilde{d} \in \mathbb{C}[p, q] - 0$ , as long as one observes that  $\mathbb{C}[p, q, x][1/v] = \mathbb{C}[x, y][1/v]$ . Indeed,  $d \in \mathbb{C}[p, q, x][1/v] = \mathbb{C}[x, y][1/v]$ , hence  $d = \tilde{d}/v^l$  for some  $\tilde{d} \in \mathbb{C}[x, y] - 0$  and  $l \geq 0$ , etc.).

So  $d = \tilde{d}/v^l$ , with  $\tilde{d} \in \mathbb{C}[p, q] - 0$ . Let  $D = d(U_1, U_2) = \tilde{d}(U_1, U_2)/v^l(U_1, U_2) = \tilde{d}(U_1, U_2)/V^l$ . Clearly,  $\beta(D) = d$ .

For convenience, multiply the above equation  $\beta(\partial H / \partial U_3) = \lambda/d$  by  $d$  and get  $d\beta(\partial H / \partial U_3) = \lambda$ . Then  $\beta(D)\beta(\partial H / \partial U_3) = \lambda$ , so  $\beta(D\partial H / \partial U_3) = \beta(\lambda)$ . Therefore,  $D\partial H / \partial U_3 - \lambda$  is in the kernel of  $\beta$ .

We have seen that the kernel of  $\beta$  is a principal ideal of  $\mathbb{C}[U_1, U_2, U_3][1/V]$ , generated by  $H \in \mathbb{C}[U_1, U_2, U_3]$ , hence there exist  $S \in \mathbb{C}[U_1, U_2, U_3]$  and  $t \geq 0$  such that  $D\partial H / \partial U_3 - \lambda = (S/V^t)H$ . Replace  $D$  by  $\tilde{d}(U_1, U_2)/V^l$  and get,  $(\tilde{d}(U_1, U_2)/V^l)\partial H / \partial U_3 - \lambda = (SH)/V^t$ . Multiply both sides by  $V^{l+t}$  and get,  $V^t\tilde{d}(U_1, U_2)\partial H / \partial U_3 - \lambda V^{l+t} = V^l(SH)$ .

Now, as promised above, we use the fact that the  $U_3$ -degree of  $V$  is zero: The  $U_3$ -degree of the right side is at least  $r$  (= that of  $H$ , which is exactly  $r$ , plus that of  $S$ , which is  $\geq 0$ ), while the  $U_3$ -degree of the left side is exactly  $r-1$  (= that of  $\partial H / \partial U_3$ ).

It follows that  $S = 0$  and  $r - 1 = 0$ , so  $r = 1$  and  $H = H_1(U_1, U_2)U_3 + H_0(U_1, U_2)$ . Apply  $\beta$  and get  $0 = H_1(p, q)x + H_0(p, q)$ , so  $x = -H_0(p, q)/H_1(p, q) \in \mathbb{C}(p, q)$ . By Wang's intersection theorem,  $x \in \mathbb{C}[p, q]$ . Then obviously,  $\mathbb{C}[p, q][y] = \mathbb{C}[x, y]$ . Finally, Formanek's automorphism theorem implies that  $\mathbb{C}[p, q] = \mathbb{C}[x, y]$ , namely  $f$  is an automorphism.  $\square$

All the arguments and known results we use do not depend on Keller's theorem, hence we have a new proof of Keller's theorem in dimension two:

**Theorem 3.2** (Keller's theorem). *Let  $f : \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y]$  be a  $\mathbb{C}$ -algebra endomorphism that satisfies  $\text{Jac}(p, q) \in \mathbb{C}^*$ . If  $\mathbb{C}(p, q) = \mathbb{C}(x, y)$ , then  $f$  is an automorphism.*

*Proof.* The group of invertible elements of  $\mathbb{C}[p, q, x][1/v] \subset \mathbb{C}(x, y)$  is contained in  $\mathbb{C}(x, y) - 0 = \mathbb{C}(p, q) - 0$ . Now apply Theorem 3.1.  $\square$

Notice that the converse of Theorem 3.1 is trivially true: If  $f$  is an automorphism, then  $\mathbb{C}[p, q] = \mathbb{C}[x, y]$ , so  $\mathbb{C}(p, q) = \mathbb{C}(x, y)$ , hence the group of invertible elements of  $\mathbb{C}[p, q, x][1/v] \subset \mathbb{C}(x, y)$  is contained in  $\mathbb{C}(x, y) - 0 = \mathbb{C}(p, q) - 0$ .

Another argument: If  $f$  is an automorphism, then we can take  $u = y$  and  $v = 1$ . Then  $\mathbb{C}[p, q, x][1/v] = \mathbb{C}[x, y]$ , and its group of invertible elements is  $\mathbb{C}^*$ , which is contained in  $\mathbb{C}(p, q) - 0$ .

Therefore, the condition in Keller's theorem is equivalent to our condition, not just implies our condition:

**Proposition 3.3.** *TFAE:*

- (i)  $f$  is an automorphism, i.e.  $\mathbb{C}[p, q] = \mathbb{C}[x, y]$ .
- (ii)  $f$  is birational, i.e.  $\mathbb{C}(p, q) = \mathbb{C}(x, y)$ .
- (iii) The group of invertible elements of  $\mathbb{C}[p, q, x][1/v]$  is contained in  $\mathbb{C}(p, q) - 0$ .

We do not know how to show directly that (iii) implies (ii).

## 4 Further discussion

We wish to bring some related ideas.

**First idea:** We have already mentioned in the Preliminaries that Formanek remarks that  $\mathbb{C}(p, q, w) = \mathbb{C}(x, y)$ , where  $w$  is the image of  $x$  under any automorphism of  $\mathbb{C}[x, y]$ . Therefore, we can obtain similar theorems to Theorem 3.1 with  $x$  replaced by any image of  $x$  under an automorphism of  $\mathbb{C}[x, y]$ .

More elaborately, take any automorphism  $g : \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y]$  and denote  $g_1 := g(x)$  and  $g_2 := g(y)$ .

We have  $\mathbb{C}(p, q, g_1) = \mathbb{C}(x, y) = \mathbb{C}(g_1, g_2)$ ; the first equality follows from Formanek's remark, while the second equality trivially follows from  $\mathbb{C}[x, y] = \mathbb{C}[g_1, g_2]$ . Then,  $g_2 \in \mathbb{C}(p, q)(g_1) = \mathbb{C}(p, q)[g_1]$ , because  $g_1$  is algebraic over  $\mathbb{C}(p, q)$ . It is easy to obtain  $g_2 = u_g/v_g$ , where  $u_g \in \mathbb{C}[p, q, g_1] - 0$  and  $v_g \in \mathbb{C}[p, q] - 0$ .

**Theorem 4.1.** *If the group of invertible elements of  $\mathbb{C}[p, q, g_1][1/v_g]$  is contained in  $\mathbb{C}(p, q) - 0$ , then  $f$  is an automorphism.*

*Proof.* In the proof of Theorem 3.1 replace  $x$  and  $y$  by  $g_1$  and  $g_2$ , do the appropriate adjustments, and get a proof for the new theorem. Notice that now, instead of considering  $p$  and  $q$  as functions of  $x$  and  $y$ , one has to consider  $p$  and  $q$  as functions of  $g_1$  and  $g_2$ .  $\square$

**Second idea:** For  $v$  as in Corollary 2.3 write  $v = v_1 \cdots v_m$ , where  $v_1, \dots, v_m \in \mathbb{C}[p, q]$  are irreducible elements of  $\mathbb{C}[p, q]$ . There are two options, either one (or

more) of the  $v_j$ 's becomes reducible in  $\mathbb{C}[x, y]$  or all the  $v_j$ 's remain irreducible in  $\mathbb{C}[x, y]$ .

If one (or more) of the  $v_j$ 's becomes reducible in  $\mathbb{C}[x, y]$ , then it is possible to show that our condition of Theorem 3.1 is not satisfied, and hence  $f$  is not an automorphism: Assume that  $v_1 = w_1 \cdots w_l$ , where  $w_1, \dots, w_l \in \mathbb{C}[x, y]$  are irreducible in  $\mathbb{C}[x, y]$ ,  $l > 1$ . It is not difficult to see (use Wang's intersection theorem) that at least two factors are in  $\mathbb{C}[x, y] - \mathbb{C}[p, q]$ , w.l.o.g  $w_1$  and  $w_2$ . We claim that  $w_1$  is invertible in  $\mathbb{C}[p, q, x][1/v] = \mathbb{C}[x, y][1/v]$ . Indeed,  $1 = v/v = v_1 \cdots v_m/v = w_1 w_2 \cdots w_l v_2 \cdots v_m/v = w_1(w_2 \cdots w_l v_2 \cdots v_m/v)$ .

Clearly,  $w_1 \notin \mathbb{C}(p, q)$ , because otherwise,  $w_1 \in \mathbb{C}(p, q) \cap \mathbb{C}[x, y] = \mathbb{C}[p, q]$ , but  $w_1 \in \mathbb{C}[x, y] - \mathbb{C}[p, q]$ .

Actually, if one (or more) of the  $v_j$ 's becomes reducible in  $\mathbb{C}[x, y]$ , then it is immediate that  $f$  is not an automorphism, since an automorphism satisfies  $\mathbb{C}[p, q] = \mathbb{C}[x, y]$ , so trivially every irreducible element of  $\mathbb{C}[p, q]$  is an irreducible element of  $\mathbb{C}[x, y]$ .

Next, if all the  $v_j$ 's remain irreducible in  $\mathbb{C}[x, y]$ , then our condition of Theorem 3.1 is satisfied:

**Theorem 4.2** (A special case of the main theorem). *If  $v_1, \dots, v_m$  remain irreducible in  $\mathbb{C}[x, y]$ , then  $f$  is an automorphism.*

Of course, since  $\mathbb{C}[p, q] \subset \mathbb{C}[x, y]$  is a UFD, every irreducible element of  $\mathbb{C}[p, q]$  ( $\mathbb{C}[x, y]$ ) is prime.

*Proof.* By assumption,  $v_1, \dots, v_m \in \mathbb{C}[p, q]$  are irreducible elements of  $\mathbb{C}[x, y]$ , hence,  $v_1, \dots, v_m$  are prime elements of  $\mathbb{C}[x, y]$ .

Claim: The condition of Theorem 3.1 is satisfied.

Proof of claim: Let  $a \in \mathbb{C}[p, q, x][1/v] = \mathbb{C}[x, y][1/v]$  be an invertible element, so there exists  $b \in \mathbb{C}[p, q, x][1/v] = \mathbb{C}[x, y][1/v]$  such that  $ab = 1$ . We can write  $a = r/v^k$  and  $b = s/v^l$ , for some  $r, s \in \mathbb{C}[x, y] - 0$  and  $k, l \geq 0$ . Then  $ab = 1$  becomes  $rs = v^{k+l} = (v_1 \cdots v_m)^{k+l}$ . Since  $v_1, \dots, v_m$  are prime elements of  $\mathbb{C}[x, y]$ , we obtain that  $r = v_1^{\alpha_1} \cdots v_m^{\alpha_m}$  and  $s = v_1^{\beta_1} \cdots v_m^{\beta_m}$ , where  $\alpha_j + \beta_j = k+l$ ,  $1 \leq j \leq m$ . Therefore,  $r, s \in \mathbb{C}[p, q] - 0$ , so  $a = r/v^k \in \mathbb{C}(p, q) - 0$ , and we are done.  $\square$

Notice that in Theorem 4.2 we demand that each of the irreducible factors  $v_1, \dots, v_m \in \mathbb{C}[p, q]$  of  $v$  remain irreducible in  $\mathbb{C}[x, y]$ , but we do not demand that other irreducible elements of  $\mathbb{C}[p, q]$  remain irreducible in  $\mathbb{C}[x, y]$ .

If one demands that every irreducible element of  $\mathbb{C}[p, q]$  remains irreducible in  $\mathbb{C}[x, y]$ , then, without relying on Theorem 3.1, one can get that  $f$  is an automorphism, thanks to the result [8, Lemma 3.2] of Jedrzejewicz and Zieliński.

Their result says the following: Let  $A$  be a UFD. Let  $R$  be a subring of  $A$  such that  $R^* = A^*$ . The following conditions are equivalent:

- (i) Every irreducible element of  $R$  remains irreducible in  $A$ .
- (ii)  $R$  is factorially closed in  $A$ .

(Recall that a sub-ring  $R$  of a ring  $A$  is called factorially closed in  $A$  if whenever  $a_1, a_2 \in A$  satisfy  $a_1 a_2 \in R - 0$ , then  $a_1, a_2 \in R$ ).

In [8, Lemma 3.2] take  $A = \mathbb{C}[x, y]$ ,  $R = \mathbb{C}[p, q]$ ; since we now assume that every irreducible element of  $\mathbb{C}[p, q]$  remains irreducible in  $\mathbb{C}[x, y]$ , we obtain that  $\mathbb{C}[p, q]$  is factorially closed in  $\mathbb{C}[x, y]$ , and we are done by the following easy lemma:

**Lemma 4.3.** *If  $\mathbb{C}[p, q]$  is factorially closed in  $\mathbb{C}[x, y]$ , then  $f$  is an automorphism.*

*Proof.* Let  $H$  be as in the proof of Theorem 3.1, and denote  $h_j := H_j(p, q)$ ,  $0 \leq j \leq r$ . Obviously,  $h_0 \neq 0$  by the minimality of  $r$ .

We have  $x(h_r x^{r-1} + h_{r-1} x^{r-2} + \dots + h_1) = -h_0 \in \mathbb{C}[p, q] - 0$ . By assumption  $\mathbb{C}[p, q]$  is factorially closed in  $\mathbb{C}[x, y]$ , hence  $x \in \mathbb{C}[p, q]$  (and  $h_r x^{r-1} + h_{r-1} x^{r-2} + \dots + h_1 \in \mathbb{C}[p, q]$ ).

Then  $\mathbb{C}[p, q, y] = \mathbb{C}[x, y]$ , and  $f$  is an automorphism by Formanek's automorphism theorem.  $\square$

Notice that in the proof of Lemma 4.3,  $h_r x^{r-1} + h_{r-1} x^{r-2} + \dots + h_1 \in \mathbb{C}[p, q]$  also yields that  $f$  is an automorphism, because by the minimality of  $r$ , we must have  $r = 1$ , so  $h_1 x + h_0 = 0$ . Then  $x = -h_0/h_1 \in \mathbb{C}(p, q)$ , and by Wang's intersection theorem,  $x \in \mathbb{C}[p, q]$ , etc.

**Third idea:** Notations as in the second idea, another special case is when all the  $v_j$ 's are primes in  $\mathbb{C}[p, q, x]$ ; this special case is dealt with in [13]: It is shown in [13, Theorem 2.2] that if all the  $v_j$ 's are primes in  $\mathbb{C}[p, q, x]$ , then  $\mathbb{C}[p, q, x]$  is a UFD, and it is shown in [13, Theorem 2.1] that if  $\mathbb{C}[p, q, x]$  is a UFD, then  $f$  is an automorphism.

It is not yet clear to us what happens in the more general case when all the  $v_j$ 's are irreducibles in  $\mathbb{C}[p, q, x]$ . It may happen that some (or all) of the  $v_j$ 's are not primes in  $\mathbb{C}[p, q, x]$ , since we just know that  $\mathbb{C}[p, q, x]$  is an integral domain (if we knew it is a UFD, then  $f$  is an automorphism by [13, Theorem 2.1]).

**Fourth idea:**

We do not know if a similar result to Theorem 3.1 holds in higher dimensions. Even if the answer is positive, the proof should be somewhat different from the proof of the two-dimensional case. For example, already in the three-dimensional case some problems may arise when trying to generalize the proof of the two-dimensional case:

Let  $f : \mathbb{C}[x, y, z] \rightarrow \mathbb{C}[x, y, z]$  be a  $\mathbb{C}$ -algebra endomorphism having an invertible Jacobian. Denote  $p := f(x)$ ,  $q := f(y)$ ,  $r := f(z)$ .

It is not difficult to generalize Corollary 2.3:

**Corollary 4.4.** *There exist  $u \in \mathbb{C}[p, q, r, x, y] - 0$  and  $v \in \mathbb{C}[p, q, r] - 0$  such that  $z = u/v$ .*

*Proof.* By Formanek's field of fractions theorem,  $\mathbb{C}(p, q, r, x, y) = \mathbb{C}(x, y, z)$ .

Since  $x$  and  $y$  are algebraic over  $\mathbb{C}(p, q, r)$ , a generalization of [14, Remark 4.7] implies that  $\mathbb{C}(p, q, r)[x, y] = \mathbb{C}(p, q, r)(x, y)$ . Then,  $\mathbb{C}(p, q, r)[x, y] = \mathbb{C}(x, y, z) \ni z$ . From this it is not difficult to obtain that  $z = u/v$ , where  $u \in \mathbb{C}[p, q, r, x, y] - 0$  and  $v \in \mathbb{C}[p, q, r] - 0$ .  $\square$

Define:  $\alpha : \mathbb{C}[U_1, U_2, U_3, U_4, U_5] \rightarrow \mathbb{C}[p, q, r, x, y]$  by  $\alpha(U_1) := p$ ,  $\alpha(U_2) := q$ ,  $\alpha(U_3) := r$ ,  $\alpha(U_4) := x$ ,  $\alpha(U_5) := y$ . Clearly,  $\alpha$  is surjective.

We can define  $\beta : \mathbb{C}[U_1, U_2, U_3, U_4, U_5][1/V] \rightarrow \mathbb{C}[p, q, r, x, y][1/v]$ . It is clear that  $z \in \mathbb{C}[p, q, r, x, y][1/v]$ .

The kernel of  $\alpha$  is a height two prime ideal; indeed,  $\mathbb{C}[U_1, U_2, U_3, U_4, U_5]$  is of Krull dimension 5 and  $\mathbb{C}[p, q, r, x, y]$  is of Krull dimension 3, hence the kernel of  $\alpha$  is of height two.

From Krull's principal ideal theorem [3, Theorem 8.42], the kernel of  $\alpha$  is generated by at least two elements.

Assume for the moment that the kernel of  $\alpha$  is generated by exactly two elements, hence the kernel of  $\beta$  is generated by the same two elements. The matrix equation (2) in Formanek's proof will involve the product of a  $5 \times 5$  matrix with a  $5 \times 5$  matrix, but Cramer's Rule seems not to help here. We do not know yet if it is possible to overcome this problem.

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